ALGEBRA COMPREHENSIVE EXAMINATION

Spring 2024

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Directions: *Answer 5 questions only.* You must answer *at least one* from each of linear algebra, groups, and synthesis. Indicate CLEARLY which problems you want us to grade—otherwise, we will select which ones to grade, and they may not be the ones that you want us to grade. Be sure to show enough work that your answers are adequately supported.

Notation: $\mathbb{R}$ is the set of real numbers; $\mathbb{Z}$ is the set of integers; $\mathbb{C}$ is the set of complex numbers; $GL(V)$ is the group of invertible linear maps from a vector space V to itself; $GL_n(F)$ is the group of all invertible $n \times n$ matrices with entries in the field F; $SO_n(F)$ is the subgroup of $GL_n(F)$ consisting of all matrices that are orthogonal and have determinant one.

Linear Algebra

(1) Let $T : V \to V$ be a linear transformation from a real vector space V to itself. Let u_1 and u_2 be nonzero vectors in V such that $T(u_1) = u_1$ and $T(u_2) = 2u_2$. Show that $\{u_1, u_2\}$ is linearly independent.

(2) (a) Prove that every matrix with entries from $\mathbb{C}$ has at least one eigenvalue in $\mathbb{C}$.
 (b) Give an example of a matrix with entries from $\mathbb{R}$ that does not have an eigenvalue in $\mathbb{R}$.

(3) Let $V = M_n(\mathbb{R})$ be the vector space of $n \times n$ real matrices. Let I be the identity $n \times n$ matrix. Prove that for all matrices $A, B \in V$ the equation $A \cdot B - B \cdot A = I$ is impossible.

Groups

(1) Let H and K be subgroups of a group G such that $G = HK$. Show that the following are equivalent:
 (a) $H \cap K = \{e\}$
 (b) Each element of $g \in G$ can be written uniquely in the form $g = hk$ with $h \in H$ and $k \in K$.

(2) (a) Prove or disprove: For every finite group G, every element $g \in G$ has finite order.
 (b) Prove or disprove: For every infinite group G, every non-identity element $g \in G$ has infinite order.

(3) Let G be a group. Recall that an *automorphism* of G is an isomorphism from G to G.
 (a) Fix $a \in G$, and define a map $\phi_a \colon G \to G$ by $\phi_a(g) = aga^{-1}$. Prove that ϕ_a is an automorphism of G.
 (b) Suppose H is a subgroup of G with the property that, for every automorphism α of G, we have $\alpha(H) = H$. Prove that H is a normal subgroup of G.

Synthesis

(1) Let F be a field, and define $F^* = F \setminus \{0\}$. It is a fact (you do not have to prove) that F^* is a group under multiplication. Let V be a finite-dimensional vector space over F. For each $c \in F^*$, define $\phi_c \colon V \to V$ by

$$\phi_c(v) = c \cdot v$$

(i.e., ϕ_c is scalar multiplication by c).

(a) For fixed c, prove that ϕ_c is an invertible linear map.

(b) Prove that the map $\Phi \colon F^* \to GL(V)$ given by $\Phi(c) = \phi_c$ is a group homomorphism.

(2) Let $G = \left\{ \begin{pmatrix} 1+n & -n \\ n & 1-n \end{pmatrix} \mid n \in \mathbb{Z} \right\}$.

(a) Prove that G is group under matrix multiplication.

(b) Prove that G and $(\mathbb{Z}, +)$ are isomorphic.

(3) Let G be the subgroup of $GL_3(\mathbb{R})$ generated by the matrices

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) Is G contained in $SO_3(\mathbb{R})$? Explain.

(b) Is G abelian? Explain.

(c) Find an element of G with order two.

(d) Explain why G must have order 12 or greater.